University of Zurich

Department of Economics

Working Paper Series

ISSN 1664-7041 (print)
ISSN 1664-705X (online)

Working Paper No. 222

Diffusion of Behavior in Dynamic Networks

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Revised version, May 2017

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Abstract

We analyze binary choice models in communication networks, in which both, the formation of links in the network as well as the action choices are endogenous. We provide a complete characterization of the equilibrium action choices and networks, where agents choose their strategies – actions and links – according to a perturbed best response update rule. We show that a threshold exists in the linking cost and the conformity parameter, giving rise to either a sparse or a densely connected communication network. Moreover, we show how the theoretical model can be efficiently estimated using cross sectional data on agents choices and their network of interactions.

Key words: diffusion, technology adoption, networks, riots, protests, Arab spring
JEL: D74, D72, D71, D83, C72

1. Introduction

In this paper we study the tradeoff agents are facing between their idiosyncratic preferences for adopting a certain action and the social influence from their peers [cf. Young, 1998]. The set of relevant peers can be represented as the neighbors in a network of interactions between the agents. The network can change over time as agents are choosing dynamically with whom to interact. We thus study a binary choice model with network externalities, where both, the adoption decision as well as the network are endogenous, and depend on each other.

Such binary choice models can be applied to various contexts ranging from the adoption of a new technology, the diffusion of innovations, the consumption of goods with externalities, smoking behavior, voting, opinion formation, to the participation decision in rots or protests [cf. e.g. Acemoglu and Ozdaglar, 2011; Conley and Udry, 2010]. In the example of a protest, agents have to choose between participating or not in a protest. Such protests can be successful if sufficiently many people participate [Barberà and Jackson, 2016; Granovetter, 1978]. But they can be disastrous for participants if they fail. This creates a setting where communication can be useful to coordinate expectations and actions. Communication and information exchange is often bilateral and hence can be modelled as a network, in particular, when the observability of other agents' behavior is restricted to a subset of the agents in the population. We model these networks as a game of incomplete information in which each person, given his local knowledge from the contacts in the network, decides whether to participate in a protest [cf. Myers, 2000]. Moreover, the decision of the agents with whom to interact is itself endogenous and depends, in turn, on the action of the agents.

As the payoff of an agent in our model increases with the number of links he maintains with other agents who choose the same action, we incorporate homophily, i.e. the tendency of agents to associate disproportionately with those having similar characteristics and behavior [Currarini

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This version: May 11, 2017, First version: February 27, 2015

et al., 2009; Golub and Jackson, 2012; McPherson et al., 2001]. However, we not only account for the fact that similar agents tend to form links with each other, but also that agents become more similar with those they are connected to.

Our model allows us to investigate the existence of a critical mass in binary choices models with externalities, and how such a threshold depends on the network structure. We show that a threshold exists in the linking cost and the conformity parameter (measuring the peer influence), giving rise to either a sparse or a densely connected communication network.

Moreover, our framework allows us to identify influential spreaders in a network ("key players") [cf. e.g. Ballester et al., 2006; Banerjee et al., 2014; González-Bailón et al., 2011], and whether the success of a protest depends on the existence of such influential individuals, or whether a successful protest is an emerging phenomenon, in which the network structure, respectively, the dynamic interaction pattern where link formation is costly, of ex ante identical individuals determines the outcome.

Finally, we use the complete equilibrium characterization of our model to derive a simple and efficient estimation procedure that can be applied to cross sectional data of agents' choices and their network of interactions.

Related Literature There exists a growing literature of binary choice models with (exogenous) externalities pioneered by Brock and Durlauf [2001] with various generalizations and applications discussed in Blume et al. [2011]. In particular, Krauth [2006] analyzes the model by Brock and Durlauf [2001] on a specific network structure where agents are arranged in a cycle, and shows that this can give rise to equilibrium multiplicity. Incomplete information in the model by Brock and Durlauf [2001] is introduced in de Paula and Tang [2012] and Lee et al. [2014] generalize this incomplete information framework to arbitrary, fixed network structures. While these papers assume an exogenous network structure, we analyze a binary choice model with an endogenous network structure.

There also exist numerous theoretical models for the spreading of epidemics in networks [cf. Jackson and Rogers, 2007; Newman, 2010; Van Mieghem, 2011; Van Mieghem et al., 2009]. In these models a threshold for the probability with which information is passed on between neighbors exists. Above the threshold information spreads through the entire network, and thus becomes epidemic. Similarly to these works we investigate the existence of such thresholds depending on the parameters of the model, however, here the focus is on game-theoretic models that are based on the notion of utility maximization rather than exposure. The basic hypothesis is that, when adopting a behavior, each individual makes a rational choice to maximize his or her payoff in a coordination game.

Further, there exists a large literature that analyzes how social influence can affect the behavior of agents when the choice variable of an agent lies in a continuum [Lorenz, 2007]. There also exist various papers studying communication networks where agents can learn through information that is passed along the links in a network about a noisy signal of some state variable Acemoglu et al. [2014]; Golub and Jackson [2012]; Jackson and Golub [2010].[1] Differently to this literature, we consider binary choice variables and make the communication network endogenous.

Only few papers investigate the actual coevolution of networks and behavior [cf. Ehrhardt et al., 2006a,b; Fosco et al., 2010; Gleeson, 2013; Gross et al., 2006; König et al., 2014; Staudigl, 2011]. Often these models assume that links are created or removed according to some fixed, exogenous rates, while we base the linking decisions on a myopic, payoff maximizing rationale.

[1] Acemoglu et al. [2014] also consider the endogenous formation of the communication network.

A similar approach is taken in Hsie et al. [2012], but there the choices of agents are continuous, which also requires a different approach to characterize the equilibrium, and consequently their paper applies to a different context. Moreover, a similar model with binary variables has been investigated in Biely et al. [2009], but without game theoretic micro-foundations, and without allowing for heterogeneity among agents. Finally, Badev [2013] studies a similar binary choice model as we do here, and applies it to smoking behavior in friendship networks. However, this paper does not provide an explicit equilibrium characterization as we do here (nor comparative statics results), and consequently cannot use this characterization for the estimation of the model. In particular, while the estimation method in Badev [2013] requires the summation across all possible networks of size n, of which there are $2^{\binom{n}{2}}$, no such computational burden is required for our estimation method.

2. The Model

Let the strategy of each agent $i \in \mathcal{N} = \{1, \ldots, n\}$ be given by $s_i \in \{-1, +1\}$ indicating whether i wants to participate in the riot or not.[2] Further let $\mathbf{s} = (s_1, \ldots, s_n)^\top \in \{-1, +1\}^n$ where $\#(\{-1, +1\}^n) = 2^n$. Then the payoff of agent i is given by [cf. Brock and Durlauf, 2001][3,4]

$$\pi_i(\mathbf{s}, G) = (1-\theta)\gamma_i s_i + \theta \sum_{j=1}^{n} a_{ij} s_i s_j - \zeta d_i, \tag{1}$$

where $\gamma_i \in \{-1, +1\}$ is some idiosyncratic preference for participating in the riot, $\sum_{j=1}^{n} a_{ij} s_i s_j$ is the total group strategy choices identical to i, $a_{ij} \in \{0, 1\}$ indicates whether i and j are connected, d_i is the number of links of i in the network $G \in \mathcal{G}^n$, $\zeta > 0$ a fixed linking cost, and $\theta \in (0, 1)$ is a parameter weighting the idiosyncratic preference versus the group preference, i.e. a conformity parameter.

Regarding the payoff function introduced in Equation (1), Rosser [1999] states that:

> "In this sort of an economy [...considered by Brock and Durlauf, 2001] with interacting agents, gradual changes in the degree of interaction (or coordination) or gradual changes in the willingness of agents to change their attitudes (intensity of choice) can lead to discontinuous changes, in which suddenly agents will be moving together in some very different direction, as in the takeoff or crash of a speculative bubble or the emergence or disappearance of "animal spirits" or coordination in a Keynesian macro model. One can imagine applications to the cases of fads and information contagion and cascades, or revolutions arising from a brave individual speaking out, although such models have not yet been applied in these cases."

Next, observe that there exists a potential function associated with the payoff function introduced above [Monderer and Shapley, 1996].[5]

[2] The strategy space is similar to the Ising model and the spin-glass model with two possible spin states [cf. Grimmett, 2010; Reichl, 2004; Sherrington and Kirkpatrick, 1975].

[3] See also Blume et al. [2011]; Brock and Durlauf [2007] for additional discussion of this type of binary choice model with (exogenous) social interactions. Further, Krauth [2006] analyzes the model by Brock and Durlauf [2001] on a cycle. de Paula and Tang [2012] introduce incomplete information and Lee et al. [2014] generalize it to arbitrary, fixed network structures.

[4] Similar to Phan and Semeshenko [2008] we assume that the agents' idiosyncratic preferences are heterogeneous and deterministic. In contrast, Brock and Durlauf [2001] assume that they are heterogeneous and random.

[5] The proposition is a special case of Proposition 1 in Hsie et al. [2012].

Proposition 1. *The payoff function in Equation (1) admits a potential function*

$$\Phi(\mathbf{s}, G) = (1-\theta)\sum_{i=1}^{n}\gamma_i s_i + \frac{\theta}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}s_i s_j - m\zeta,$$

for both action and link adjustments, where m counts the number of links in the network G.

The potential function in Proposition 1 will be useful to characterize the stationary states of the stochastic process of network formation and action adjustments that we will introduce in the following section.

3. Network Formation and Action Adjustment

We endogenize the action choices and the network using a stochastic process akin to Hsie et al. [2012]. In this process the opportunities for change arrive as a Poisson process [cf. Sandholm, 2010]. To capture the fact that agents are uncertain about the behavior of their neighbors or the consequences (costs) of their actions, we introduce noise in this decision process [cf. e.g. Blume, 1993; Kandori et al., 1993].

Definition 1. *The evolution of the population of agents and the links between them is characterized by a sequence of states $(\boldsymbol{\omega}_t)_{t\in\mathbb{R}_+}$, $\omega_t \in \Omega$, where each state $\boldsymbol{\omega}_t = (\mathbf{s}_t, G_t)$ consists of a vector of agents' actions $\mathbf{s}_t \in \{-1,+1\}^n$ and a network $G_t \in \mathcal{G}^n$. In a short time interval $[t, t+\Delta t)$, $t \in \mathbb{R}_+$, one of the following events happens:*

Action adjustment *At rate $\chi \geq 0$ an agent $i \in \mathcal{N}$ is selected at random and given a revision opportunity of its current action $s_{it} \in \{-1,+1\}$. When agent i receives such a revision opportunity, he evaluates the marginal payoff from changing its current action s_{it} to s_i'. The computation of marginal payoffs is perturbed by an additive i.i.d. shock ε_{it}, so that the probability that we observe a switch from action s_{it} to s_i' is given by*

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (s_i', \mathbf{s}_{-it}, G_t)|\boldsymbol{\omega}_t = (s_{it}, \mathbf{s}_{-it}, G_t)\right)& \\ = \chi\mathbb{P}\left(\pi_i(s_i', \mathbf{s}_{-it}, G_t) - \pi_i(s_{it}, \mathbf{s}_{-it}, G_t) + \varepsilon_{it} > 0\right)\Delta t + o(\Delta t)& \\ = \chi\mathbb{P}\left(\Phi(s_i', \mathbf{s}_{-it}, G_t) - \Phi(s_{it}, \mathbf{s}_{-it}, G_t) + \varepsilon_{it} > 0\right)\Delta t + o(\Delta t).&\end{aligned}$$

where we have used the fact that $\pi_i(s_i', \mathbf{s}_{-it}, G_t) - \pi_i(q_{it}, \mathbf{s}_{-it}, G_t) = \Phi(s_i', \mathbf{s}_{-it}, G_t) - \Phi(s_{it}, \mathbf{s}_{-it}, G_t)$. In the following we will make a specific assumption on the distribution of the random shocks. In particular, we assume that these shocks are independent and identically exponentially distributed with parameter $\eta \geq 0$. We then can write[6]

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t} = (s_i', \mathbf{s}_{-it}, G_t)|\boldsymbol{\omega}_t = (s_i, \mathbf{s}_{-it}, G_t)\right) &= \chi\,\mathbb{P}\left(-\varepsilon_{it} < \Phi(s_i', \mathbf{s}_{-it}, G_t) - \Phi(s_{it}, \mathbf{s}_{-it}, G_t)\right)\Delta t + o(\Delta t) \\ &= \chi\frac{e^{\eta\Phi(s_i', \mathbf{s}_{-i}, G_t)}}{e^{\eta\Phi(s_i', \mathbf{s}_{-it}, G_t)} + e^{\eta\Phi(s_i, \mathbf{s}_{-it}, G_t)}}\Delta t + o(\Delta t),\end{aligned}$$

Link formation *With rate $\lambda \geq 0$ a pair of agetns ij which is not already connected receives an opportunity to form a link. The formation of a link depends on the marginal payoff the*

[6] Let z be i.i. logistically distributed with mean 0 and scale parameter η, i.e. $F_z(x) = \frac{e^{\eta x}}{1+e^{\eta x}}$. Consider the random variable $\varepsilon = g(z) = -z$. Since g is monotonic decreasing, and z is a continuous random variable, the distribution of ε is given by $F_\varepsilon(y) = 1 - F_z(g^{-1}(y)) = \frac{e^{\eta y}}{1+e^{\eta y}}$.

agents receive from the link plus an additive pairwise i.i.d. error term $\varepsilon_{ij,t}$. The probability that link ij is created is then given by

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t}=(\mathbf{s}_t,G_t+ij)|\boldsymbol{\omega}_{t-1}=(\mathbf{s},G_t)\right)&=\lambda\,\mathbb{P}\left(\{\pi_i(\mathbf{s}_t,G_t+ij)-\pi_i(\mathbf{s}_t,G_t)+\varepsilon_{ij,t}>0\}\right.\\&\left.\cap\{\pi_j(\mathbf{s}_t,G_t+ij)-\pi_j(\mathbf{s}_t,G_t)+\varepsilon_{ij,t}>0\}\right)\Delta t+o(\Delta t)\\&=\lambda\,\mathbb{P}\left(\Phi(\mathbf{s}_t,G_t+ij)-\Phi(\mathbf{s}_t,G_t)+\varepsilon_{ij,t}>0\right)\Delta t+o(\Delta t),\end{aligned}$$

where we have used the fact that $\pi_i(\mathbf{s}_t,G_t+ij)-\pi_i(\mathbf{s}_t,G_t)=\pi_j(\mathbf{s}_t,G_t+ij)-\pi_j(\mathbf{s}_t,G_t)=\Phi(\mathbf{s}_t,G_t+ij)-\Phi(\mathbf{s}_t,G_t)$. Assuming that the error term $\varepsilon_{ij,t}$ is independently logistically distributed, we obtain for the creation of the link ij

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t}=(\mathbf{s}_t,G_t+ij)|\boldsymbol{\omega}_t=(\mathbf{s}_t,G_t)\right)&=\lambda\,\mathbb{P}\left(-\varepsilon_{ij,t}<\Phi(\mathbf{s}_t,G_t+ij)-\Phi(\mathbf{s}_t,G_t)\right)\Delta t+o(\Delta t)\\&=\lambda\frac{e^{\eta\Phi(\mathbf{s}_t,G_t+ij)}}{e^{\eta\Phi(\mathbf{s}_t,G_t+ij)}+e^{\eta\Phi(\mathbf{s}_t,G_t)}}\Delta t+o(\Delta t).\end{aligned}\tag{2}$$

Link removal *With rate $\xi\geq 0$ a pair of linked agents i,j receives an opportunity to terminate their connection. The link is removed if at least one agent finds this profitable. The marginal payoffs from removing the link ij are perturbed by an additive pairwise i.i.d. error term $\varepsilon_{ij,t}$. The probability that the link ij is removed is then given by*

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t}=(\mathbf{s}_t,G_t-ij)|\boldsymbol{\omega}_t=(\mathbf{s},G_t)\right)&=\xi\,\mathbb{P}\left(\{\pi_i(\mathbf{s}_t,G_t-ij)-\pi_i(\mathbf{s}_t,G_t)+\varepsilon_{ij,t}>0\}\right.\\&\left.\cup\{\pi_j(\mathbf{s}_t,G_t-ij)-\pi_j(\mathbf{s}_t,G_t)+\varepsilon_{ij,t}>0\}\right)\Delta t+o(\Delta t)\\&=\xi\,\mathbb{P}\left(\Phi(\mathbf{s}_t,G_t-ij)-\Phi(\mathbf{s}_t,G_t)+\varepsilon_{ij,t}>0\right)\Delta t+o(\Delta t),\end{aligned}$$

where we have used the fact that $\pi_i(\mathbf{s}_t,G_t-ij)-\pi_i(\mathbf{s}_t,G_t)=\pi_j(\mathbf{s}_t,G_t-ij)-\pi_j(\mathbf{s}_t,G_t)=\Phi(\mathbf{s}_t,G_t-ij)-\Phi(\mathbf{s}_t,G_t)$. When the error term is independently logistically distributed we obtain

$$\begin{aligned}\mathbb{P}\left(\boldsymbol{\omega}_{t+\Delta t}=(\mathbf{s}_t,G_t-ij)|\boldsymbol{\omega}_t=(\mathbf{s}_t,G_t)\right)&=\xi\,\mathbb{P}\left(-\varepsilon_{ij,t}<\Phi(\mathbf{s}_t,G_t-ij)-\Phi(\mathbf{s}_t,G_t)\right)\Delta t+o(\Delta t)\\&=\xi\frac{e^{\eta\Phi(\mathbf{s}_t,G_t-ij)}}{e^{\eta\Phi(\mathbf{s}_t,G_t-ij)}+e^{\eta\Phi(\mathbf{s}_t,G_t)}}\Delta t+o(\Delta t).\end{aligned}$$

We can numerically implement the stochastic process introduced in Definition 1 using the "next reaction method" for simulating a continuous time Markov chain [cf. Anderson, 2012; Gibson and Bruck, 2000]. We will use this method throughout the paper to illustrate our theoretical predictions for various network statistics (see e.g. Figure 1).

Let $\mathcal{F}$ denote the smallest σ-algebra generated by $\boldsymbol{\sigma}\left(\boldsymbol{\omega}_t : t\in\mathbb{R}_+\right)$. The filtration is the non-decreasing family of sub-σ-algebras $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ on the measure space $(\Omega,\mathcal{F})$, with the property that $\mathcal{F}_0\subseteq\mathcal{F}_1\subseteq\cdots\subseteq\mathcal{F}_t\subseteq\cdots\subseteq\mathcal{F}$. The probability space is given by the triple $(\Omega,\mathcal{F},\mathbb{P})$, where $\mathbb{P}\colon\mathcal{F}\to[0,1]$ is the probability measure satisfying $\int_\Omega\mathbb{P}(\boldsymbol{\omega})d\mu(\boldsymbol{\omega})=1$. As we will see below the sequence of states $(\boldsymbol{\omega}_t)_{t\in\mathbb{R}_+}$, $\boldsymbol{\omega}_t\in\Omega$ induces an irreducible and positive recurrent (i.e. ergodic) time homogeneous Markov chain.

The one step transition probability matrix $\mathbf{P}(t):\Omega^2\to[0,1]$ from a state $\omega\in\Omega$ to a state $\omega'\in\Omega$ is given by $\mathbb{P}(\boldsymbol{\omega}_{t+\Delta t}=\boldsymbol{\omega}'|\mathcal{F}_t=\sigma(\boldsymbol{\omega}_0,\boldsymbol{\omega}_1,\ldots,\boldsymbol{\omega}_t=\boldsymbol{\omega}))=\mathbb{P}(\boldsymbol{\omega}_{t+\Delta t}=\boldsymbol{\omega}'|\boldsymbol{\omega}_t=\boldsymbol{\omega})=q(\boldsymbol{\omega},\boldsymbol{\omega}')\Delta t+o(\Delta t)$ if $\omega'\neq\omega$, where $q(\boldsymbol{\omega},\boldsymbol{\omega}')$ is the transition rate from state $\boldsymbol{\omega}$ to state $\boldsymbol{\omega}'$. The transition rate matrix (or infinitesimal generator) $\mathbf{Q}=(q(\boldsymbol{\omega},\boldsymbol{\omega}'))_{\boldsymbol{\omega},\boldsymbol{\omega}'\in\Omega}$ of the Markov chain is given by

$$q(\boldsymbol{\omega},\boldsymbol{\omega}') = \begin{cases} \chi\frac{e^{\eta\Phi(s,\mathbf{s}_{-i},G)}}{e^{\eta\Phi(s,\mathbf{s}_{-i},G)}+e^{\eta\Phi(s',\mathbf{s}_{-i},G)}} & \text{if } \boldsymbol{\omega}'=(s_i',\mathbf{s}_{-i},G) \text{ and } \boldsymbol{\omega}=(\mathbf{s},G), \\ \lambda\frac{e^{\eta\Phi(\mathbf{s},G+ij)}}{e^{\eta\Phi(\mathbf{s},G+ij)}+e^{\eta\Phi(\mathbf{s},G)}} & \text{if } \boldsymbol{\omega}'=(\mathbf{s},G+ij) \text{ and } \boldsymbol{\omega}=(\mathbf{s},G), \\ \xi\frac{e^{\eta\Phi(\mathbf{s},G-ij)}}{e^{\eta\Phi(\mathbf{s},G-ij)}+e^{\eta\Phi(\mathbf{s},G)}} & \text{if } \boldsymbol{\omega}'=(\mathbf{s},G-ij) \text{ and } \boldsymbol{\omega}=(\mathbf{s},G), \\ -\sum_{\boldsymbol{\omega}'\neq\boldsymbol{\omega}} q(\boldsymbol{\omega},\boldsymbol{\omega}') & \text{if } \boldsymbol{\omega}'=\boldsymbol{\omega}, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

satisfying the Chapman-Kolmogorov forward equation $\frac{d}{dt}\mathbf{P}(t)=\mathbf{P}(t)\mathbf{s}$ so that $\mathbf{P}(t)=\mathbf{I}+\mathbf{Q}\Delta t+o(\Delta t)$. As the Markov chain is time homogeneous, the transition rates are independent of time. The stationary distribution $\mu^\eta:\Omega\to[0,1]$ is then the solution to $\boldsymbol{\mu}^\eta\mathbf{P}=\boldsymbol{\mu}^\eta$, or equivalently $\boldsymbol{\mu}^\eta\mathbf{Q}=\mathbf{0}$ [cf. e.g. Norris, 1998].

In the following proposition we completely characterize the equilibrium action choices and networks of the above stochastic process:

Proposition 2. *Consider a dynamic process $(\boldsymbol{\omega}_t)_{t\in\mathbb{R}_+}$ in which agents' payoffs are randomly perturbed with additive i.i.d. logistically distributed shocks with parameter $\eta>0$, and assume that agents choose their strategies according to a perturbed best response update rule as in Definition 1. Then this process induces an ergodic Markov chain with a unique stationary distribution μ^η defined on the measurable space $(\Omega,\mathcal{F})$ such that $\lim_{t\to\infty}\mathbb{P}(\boldsymbol{\omega}_t=(\mathbf{s},G)|\boldsymbol{\omega}_0=(\mathbf{s}_0,G_0))=\mu^\eta(\mathbf{s},G)$. The probability measure μ^η is given by*

$$\mu^\eta(\mathbf{s},G)=\frac{e^{\eta\Phi(\mathbf{s},G)}}{\sum_{G'\in\mathcal{G}^n}\sum_{\mathbf{s}'\in\{-1,+1\}^n}e^{\eta\Phi(\mathbf{s}',G')}}. \tag{4}$$

Proposition 2 allows us to characterize the equilibria in a fully dynamic framework, where not only the strategies s_i but also the links a_{ij} are endogenous.

We next compute the *partition function* [cf. e.g. Grimmett, 2010; Wainwright and Jordan, 2008], which appears as the denominator in Equation (4), explicitly. We have that[7]

$$\begin{aligned} \mathscr{Z}^\eta &\equiv \sum_{G\in\mathcal{G}^n}\sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\Phi(\mathbf{s},G)} \\ &= \sum_{\mathbf{s}\in\{-1,+1\}^n}\sum_{G\in\mathcal{G}^n} e^{\eta\left((1-\theta)\sum_{i=1}^n\gamma_i s_i+\sum_{i=1}^n\sum_{j=i+1}^n a_{ij}(\theta s_i s_j-\zeta)\right)} \\ &= \sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta(1-\theta)\sum_{i=1}^n\gamma_i s_i}\sum_{G\in\mathcal{G}^n} e^{\eta\sum_{i=1}^n\sum_{j=i+1}^n a_{ij}(\theta s_i s_j-\zeta)} \\ &= \sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta(1-\theta)\sum_{i=1}^n\gamma_i s_i}\prod_{i=1}^n\prod_{j=i+1}^n\left(1+e^{\eta(\theta s_i s_j-\zeta)}\right), \end{aligned} \tag{5}$$

[7]Note that when the network is exogenous (i.e. when $\xi=\lambda=0$) then in the limit of $\eta\to\infty$ the sum over all configurations $\mathbf{s}\in\{-1,+1\}^n$ is equivalent to summing over all max cuts of the underlying graph, whose enumeration is an NP hard problem (cf. A. Montanari, "Inference in Graphical Models", Stanford University, lecture notes, 2012).

where we have used the fact that

$$\sum_{G\in\mathcal{G}^n} e^{\eta\sum_{i<j}^n a_{ij}\sigma_{ij}} = \prod_{i=1}^{n}\prod_{j=i+1}^{n}\left(1+e^{\eta\sigma_{ij}}\right), \tag{6}$$

for some $\sigma_{ij}=\sigma_{ji}$. Introducing the *Hamiltonian* [cf. e.g. Grimmett, 2010]

$$\mathscr{H}^\eta(\mathbf{s}) \equiv \sum_{i=1}^{n}\left((1-\theta)\gamma_i s_i + \sum_{j=i+1}^{n}\left(\frac{1}{\eta}\ln\left(1+e^{\eta(\theta s_i s_j-\zeta)}\right)\right)\right), \tag{7}$$

we can write the partition function as follows

$$\mathscr{Z}^\eta = \sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\mathscr{H}^\eta(\mathbf{s})}. \tag{8}$$

We can use Equation (6) also to compute the marginal distribution

$$\begin{aligned}\mu^\eta(\mathbf{s}) &= \frac{1}{\mathscr{Z}^\eta}\sum_{G\in\mathcal{G}^n} e^{\eta\Phi(\mathbf{s},G)}\\ &= \frac{1}{\mathscr{Z}^\eta_n} e^{\eta(1-\theta)\sum_{i=1}^n\gamma_i s_i}\prod_{i=1}^{n}\prod_{j=i+1}^{n}\left(1+e^{\eta(\theta s_i s_j-\zeta)}\right)\\ &= \frac{1}{\mathscr{Z}^\eta} e^{\eta\mathscr{H}^\eta(\mathbf{s})},\end{aligned} \tag{9}$$

where $\mathscr{H}^\eta(\mathbf{s})$ has been defined in Equation (7). With the marginal distribution from Equation (9) we can write the conditional distribution as

$$\begin{aligned}\mu^\eta(G|\mathbf{s}) = \frac{\mu^\eta(\mathbf{s},G)}{\mu^\eta(\mathbf{s})} &= \frac{e^{\eta\left((1-\theta)\sum_{i=1}^n\gamma_i s_i+\frac{\theta}{2}\sum_{i=1}^n\sum_{j=1}^n a_{ij}s_i s_j-m\zeta\right)}}{e^{\eta(1-\theta)\sum_{i=1}^n\gamma_i s_i}\prod_{i=1}^n\prod_{j=i+1}^n\left(1+e^{\eta(\theta s_i s_j-\zeta)}\right)}\\ &= \frac{e^{\eta\sum_{i<j}^n a_{ij}(\theta s_i s_j-\zeta)}}{\prod_{i=1}^n\prod_{j=i+1}^n\left(1+e^{\eta(\theta s_i s_j-\zeta)}\right)}\\ &= \prod_{i<j}\frac{e^{\eta a_{ij}(\theta s_i s_j-\zeta)}}{1+e^{\eta(\theta s_i s_j-\zeta)}}\\ &= \prod_{i<j}\left(\frac{e^{\eta(\theta s_i s_j-\zeta)}}{1+e^{\eta(\theta s_i s_j-\zeta)}}\right)^{a_{ij}}\left(1-\frac{e^{\eta(\theta s_i s_j-\zeta)}}{1+e^{\eta(\theta s_i s_j-\zeta)}}\right)^{1-a_{ij}}\\ &= \prod_{i<j} p_{ij}(s_i,s_j)^{a_{ij}}\left(1-p_{ij}(s_i,s_j)\right)^{1-a_{ij}}.\end{aligned} \tag{10}$$

Hence, we obtain the likelihood of an inhomogeneous random graph with link probability

$$p_{ij}(s_i,s_j) = \frac{e^{\eta(\theta s_i s_j-\zeta)}}{1+e^{\eta(\theta s_i s_j-\zeta)}}. \tag{11}$$

In the following we provide an explicit computation of the partition function introduced in Equation (5).

Proposition 3. *The partition function of Equation (5) is given by*

$$\mathscr{Z}^\eta = \sum_{k=0}^{n} \sum_{j=0}^{\min\{k,n_+\}} \binom{n_+}{j}\binom{n-n_+}{k-j} e^{\eta(1-\theta)(2k-n)} \left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}} \left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}}, \tag{12}$$

where

$$l(k,j) = \frac{n^2+(2(2j-k)-1)n+2(2j-k)^2-2(n+2(2j-k)-n_+)n_+}{2}, \tag{13}$$

and $n_+ = \#(\{\gamma_i = 1 : i = 1,\ldots,n\})$.

Note that, while the evaluation of the partition function in Equation (5) requires the computation of a sum with 2^n terms, the partition function in Equation (12) requires the evaluation of only $\frac{1}{2}(n_+ + 1)(2(n+1)-n_+) = O(n)$ terms. With Equation (12) the marginal distribution $\mu^\eta(\mathbf{s})$ in Equation (9) can then be efficiently computed.

The following proposition characterizes the expected number of links as a function of the parameters of the model.

Proposition 4. *The expected number of links in the stationary state is given by*

$$\begin{aligned}\mathbb{E}^\eta(m) = \frac{1}{\mathscr{Z}^\eta} \sum_{k=0}^{n} \sum_{j=0}^{\min\{k,n_+\}} \binom{n_+}{j}\binom{n-n_+}{k-j} \frac{1}{\eta} e^{\eta(1-\theta)(2k-n)} \\ \times \left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}} \left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}} \left(\frac{l(k,j)}{1+e^{-\eta(\theta-\zeta)}} + \frac{\binom{n}{2}-l(k,j)}{1+e^{\eta(\theta+\zeta)}}\right),\end{aligned} \tag{14}$$

where $l(k,j)$ *is defined in Equation (13),* $n_+ = \#(\{\gamma_i = 1 : i = 1,\ldots,n\})$*, and we have that* $\lim_{\eta\to\infty} \mathbb{E}^\eta(m) = 0$.

An example of the average degree $\bar{d} = 2m/n$ across different values of the linking cost $\zeta \in \{0,1,\ldots,10\}$ and the conformity parameter $\theta \in [0,1]$ can be seen in Figure 1. As expected, the average degree is decreasing with increasing linking costs ζ and increasing with increasing conformity θ.

Next we consider the average action level. We can state the following proposition.

Proposition 5. *The expected average action level,* $\bar{s} = \frac{1}{n}\sum_{i=1}^{n} s_i$*, in the stationary state is given by*

$$\begin{aligned}\mathbb{E}^\eta(\bar{s}) = \frac{1}{\mathscr{Z}^\eta} \sum_{k=0}^{n} \sum_{j=0}^{\min\{k,n_+\}} \binom{n_+}{j}\binom{n-n_+}{k-j} \frac{n+4j-2(n_+ + k)}{n} \\ \times e^{\eta(1-\theta)(2k-n)} \left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}} \left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}},\end{aligned} \tag{15}$$

where $l(k,j)$ *is defined in Equation (13) and* $n_+ = \#(\{\gamma_i = 1 : i = 1,\ldots,n\})$.

We conclude this section with a characterization of the the stationary state in the vanishing noise limit. When $\eta \to \infty$, the *stochastically stable* states in the support of μ^η are given by [Kandori et al., 1993]

$$\lim_{\eta\to\infty} \mu^\eta(\mathbf{s}, G) \begin{cases} > 0, & \text{if } \Phi(\mathbf{s}, G) \geq \Phi(\mathbf{s}', G'), \quad \forall \mathbf{s}' \in \{-1,+1\}^n, \quad G' \in \mathcal{G}^n, \\ = 0, & \text{otherwise.} \end{cases} \tag{16}$$

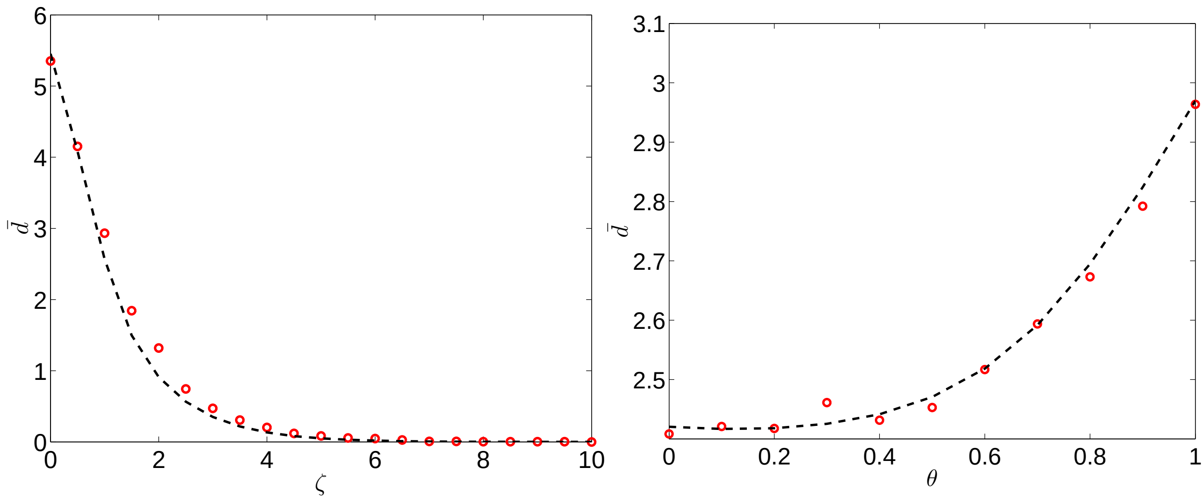

Figure 1: (Left panel) The average degree $\bar{d} = 2m/n$ across different values of the linking cost $\zeta \in \{0, 1, \ldots, 10\}$. The parameters used are $n = 10$, $n_+ = 5$, $\eta = 1$, $\lambda = \chi = \xi = 1$ and $\theta = 0.5$. (Right panel) The average degree $\bar{d}$ across different values of the conformity parameter $\theta \in [0, 1]$. The parameters used are $n = 10$, $n_+ = 5$, $\eta = 1$, $\lambda = \chi = \xi = 1$ and $\zeta = 1$. Dashed lines indicate the theoretical prediction of Equation (14) while circles indicate averages across 1000 numerical Monte Carlo simulations of the model.

The stochastically stable states are derived in the following proposition.

Proposition 6. *In the limit of $\eta \to \infty$ the stochastically stable state is given by the action profile $s_i = \gamma_i$ for all $i = 1, \ldots, n$ and when $\zeta < \theta$ a network composed of two cliques, $K_{n^+} \cup K_{n-n_+}$, of sizes n_+ and $n - n_+$, where in the first clique all agents choose the strategies $s_i = \gamma_i = +1$ and in the second clique they choose $s_i = \gamma_i = -1$ if*

$$\theta < \theta^* = \frac{n + n_+(n - n_+)\zeta - \max\{2n_+ - n, n - 2n_+\}}{n(n_+ + 1) - n_+^2 - \max\{2n_+ - n, n - 2n_+\}}, \tag{17}$$

where $n_+ = \#(\{\gamma_i = 1 : i = 1, \ldots, n\})$, while if the inequality (17) is reversed then the stochastically stable network is a complete graph K_n in which all agents $i = 1, \ldots, n$ choose $s_i = +1$ if $n_+ > \frac{n}{2}$ or $s_i = -1$ if $n_+ < \frac{n}{2}$, and the network is empty when $\zeta > \theta$ and all agents choose their idiosyncratic preference.

Proposition 6 shows that when the idiosyncratic preference is large enough (i.e. θ is small enough) in the payoff function of Equation (1) then the society is segregated in disconnected communities in which each agent is choosing the action in accordance with her idiosyncratic preference ($\gamma_i = s_i$ for all $i = 1, \ldots, n$), while if the peer effect is strong enough (the conformity parameter θ is large enough) then the society becomes completely connected and all agents choose the same action. The action chosen in the latter case is determined by the idiosyncratic preference of the majority. That is, if more agents have an idiosyncratic preference $\gamma_i = +1$ then all agents will chose $s_i = +1$, and vice versa. Finally, if linking is too costly, then all agents are isolated and choose their idiosyncratic preference.

4. Extensions

Our framework is flexible enough to allow for a number of extensions.

4.1. Local and Global Interactions

By allowing for both, local and global interactions, we can extend Equation (1) as follows

$$\pi_i(\mathbf{s}, G) = (1-\theta)\gamma_i s_i + \theta \sum_{j=1}^{n} a_{ij} s_i s_j + \rho \sum_{j=1}^{n} s_i s_j - \zeta d_i. \tag{18}$$

The potential function is then given by

$$\Phi(\mathbf{s}, G) = (1-\theta) \sum_{i=1}^{n} \gamma_i s_i + \frac{\theta}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} s_i s_j + \frac{\rho}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j - mc,$$

and the same results as in Proposition 2 hold.

4.2. Heterogeneous Linking Costs

Note that we can allow for a pair specific cost ζ_{ij} for a link between i and j so that the payoff function of agent i reads as

$$\pi_i(\mathbf{s}, G) = (1-\theta)\gamma_i s_i + \theta \sum_{j=1}^{n} a_{ij} s_i s_j - \sum_{j=1}^{n} a_{ij} \zeta_{ij}. \tag{19}$$

The potential function is then given by

$$\Phi(\mathbf{s}, G) = (1-\theta) \sum_{i=1}^{n} \gamma_i s_i + \frac{\theta}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} s_i s_j - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \zeta_{ij},$$

and the same results as in Proposition 2 hold.

4.3. Adoption Costs

By introducing an adoption cost $c \geq 0$ for choosing action s_i (e.g. rioting), we can extend Equation (1) as follows

$$\pi_i(\mathbf{s}, G) = (1-\theta)\gamma_i s_i - c s_i + \theta \sum_{j=1}^{n} a_{ij} s_i s_j - \zeta d_i. \tag{20}$$

The potential function is then given by

$$\Phi(\mathbf{s}, G) = (1-\theta) \sum_{i=1}^{n} \gamma_i s_i - c \sum_{i=1}^{n} s_i + \frac{\theta}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} s_i s_j + \frac{\rho}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j - m\zeta, \tag{21}$$

and the same results as in Proposition 2 hold. Note that with the potential function in Equation (21), the average action $\bar{s} = \frac{1}{n} \sum_{i=1}^{n} s_i$ can be computed as $\mathbb{E}^{\eta}(\bar{s}) = -\frac{1}{n\eta} \frac{1}{\mathscr{Z}^{\eta}} \frac{\partial \mathscr{Z}^{\eta}}{\partial c}$.

5. Empirical Implications

5.1. Exogenous Networks

The probability of agent i choosing action $s_i = 1$ given the strategies $\mathbf{s}_{-i}$ of all other agents and the network G follows the spatial logistic regression model with corresponding log-odds ratio

$$\log\left(\frac{\mathbb{P}(s_i = 1|\mathbf{s}_{-i}, G)}{1 - \mathbb{P}(s_i = 1|\mathbf{s}_{-i}, G)}\right) = (1-\theta)\gamma_i + \theta \sum_{j=1}^{n} a_{ij} s_j. \tag{22}$$

The log-odds for choosing strategy $s_i = 1$ is increasing in γ_i if it is one, and the number of neighbors who are choosing strategy one. The above equation can be estimated to obtain the conformity parameter $\hat{\theta}$ and the idiosyncratic preference parameter $\hat{\gamma}_i$ for all $i = 1, \ldots, n$ [cf. Smirnov, 2010]. See also Benhabib et al. [2010]; Brock and Durlauf [2007]; Durlauf and Ioannides [2010].

5.2. Endogenous Networks

The probability of observing a network $G \in \mathcal{G}^n$, given an action profile $\mathbf{s} \in \{-1, +1\}^n$, is then determined by the conditional distribution (cf. Equation (10))

$$\mu^\eta(G|\mathbf{s}) = \frac{\mu^\eta(\mathbf{s}, G)}{\mu^\eta(\mathbf{s})} = \prod_{i<j}^{n} \frac{e^{\eta a_{ij}(\theta s_i s_j - \zeta)}}{1 + e^{\eta(\theta s_i s_j - \zeta)}}. \tag{23}$$

The marginal distribution of the agents' actions is given by (cf. Equations (9) and (12))

$$\mu^\eta(\mathbf{s}) = \frac{1}{\mathscr{Z}^\eta} e^{\eta \mathscr{H}^\eta(\mathbf{s})}, \tag{24}$$

It then follows that the likelihood of the network G and the action profile $\mathbf{s}$ can be written as

$$\mu^\eta(\mathbf{s}, G) = \mu^\eta(G|\mathbf{s}) \cdot \mu^\eta(\mathbf{s}) = e^{\eta \mathscr{H}^\eta(\mathbf{s}) - \ln \mathscr{Z}^\eta} \prod_{i<j}^{n} \frac{e^{\eta a_{ij}(\theta s_i s_j - \zeta)}}{1 + e^{\eta(\theta s_i s_j - \zeta)}}, \tag{25}$$

where we have inserted Equation (23) for $\mu^\eta(G|\mathbf{s})$, Equation (24) for $\mu^\eta(\mathbf{s})$, $\mathscr{H}^\eta(\mathbf{s})$ is given by Equation (7) and $\mathscr{Z}^\eta$ is given by Equation (12). The parameters $\boldsymbol{\theta}$ of the model can then be obtained via maximum likelihood, by maximizing Equation (25), and the variances from the Fisher information matrix [cf. e.g. Casella and Berger, 2001]:

$$\mathcal{I}(\boldsymbol{\theta}) = -\mathbb{E}\left[\left.\frac{\partial^2}{\partial \boldsymbol{\theta}^2} \log \mu^\eta(\mathbf{s}, G)\right| \boldsymbol{\theta}\right],$$

6. Conclusion

We have developed a model of endogenous participation decisions of agents with an idiosyncratic preference for choosing to participate and a peer effect. The set of peers influencing an agent is endogenous and depends, in turn, on the choices of the agents that he is connected to. Our model allows for a complete equilibrium characterization when both, the network of peers as well as the participation decisions are endogenous and interdependent. Our framework could be extended along several dimensions. First, one could introduce uncertainty about the behavior of linked agents. Related models for exogenous networks and a similar payoff function but with incomplete information have been studied in De Martí and Zenou [2014] de Paula and Tang [2012] and Lee et al. [2014]. Introducing uncertainty in our coordination game would then allow us to study global games in dynamic networks [cf. e.g. Angeletos et al., 2007; Angeletos and Pavan, 2007]. Second, a possible extension of the model could combine Acemoglu and Robinson's model [see Acemoglu and Robinson, 2005, pp. 99–107] of political regime determination with our network model of political mass protests [cf. Ellis and Fender, 2011].

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Appendix

A. Proofs

Proof of Proposition 1. The potential has the property that $\Phi(s_i', \mathbf{s}_{-i}, G) - \Phi(\mathbf{s}, G) = \pi_i(s_i', \mathbf{s}_{-i}, G), G) - \pi_i(\mathbf{s}, G) = (1-\theta)(s_i' - s_i) + \theta \sum_{j=1}^n a_{ij} s_j (s_i' - s_i)$ and that $\Phi(\mathbf{s}, G \pm ij) - \Phi(\mathbf{s}, G) = \pi_i(\mathbf{s}, G \pm ij) - \pi_i(\mathbf{s}, G) = \pm(\theta s_i s_j - \zeta)$. $\square$

Proof of Proposition 2. First, note that the embedded discrete time Markov chain is irreducible and aperiodic, and thus is ergodic and has a unique stationary distribution. Hence, also the continuous time Markov chain is ergodic and has a unique stationary distribution. The stationary distribution solves $\boldsymbol{\mu}^\eta \mathbf{Q} = \mathbf{0}$ with the transition rates matrix $\mathbf{Q} = (q(\boldsymbol{\omega}, \boldsymbol{\omega}'))_{\boldsymbol{\omega}, \boldsymbol{\omega}' \in \Omega}$ of Equation (3). This equation is satisfied when the probability distribution $\mu^\eta(\boldsymbol{\omega})$ satisfies the detailed balance condition [cf. e.g. Norris, 1998]

$$\mu^\eta(\boldsymbol{\omega}) q(\boldsymbol{\omega}, \boldsymbol{\omega}') = \mu^\eta(\boldsymbol{\omega}') q(\boldsymbol{\omega}', \boldsymbol{\omega}), \tag{26}$$

for all $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \Omega$. Observe that the detailed balance condition is trivially satisfied if $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}$ differ in more than one link or more than one quantity level. Hence, we consider only the case of link creation $G' = G + ij$ (and removal $G' = G - ij$) or an adjustment in quantity $s_i' \neq s_i$ for some $i \in \mathcal{N}$. For the case of link creation with a transition from $\boldsymbol{\omega} = (\mathbf{s}, G)$ to $\boldsymbol{\omega}' = (\mathbf{s}, G + ij)$ we can write the detailed balance condition as follows

$$\frac{1}{\mathscr{Z}\eta} e^{\eta(\Phi(\mathbf{s},G) - m \ln(\frac{\xi}{\lambda}))} \frac{e^{\eta\Phi(\mathbf{s},G+ij)}}{e^{\eta\Phi(\mathbf{s},G+ij)} + e^{\eta\Phi(\mathbf{s},G)}} \lambda = \frac{1}{\mathscr{Z}\eta} e^{\eta(\Phi(\mathbf{s},G+ij) - (m+1) \ln(\frac{\xi}{\lambda}))} \frac{e^{\eta\Phi(\mathbf{s},G)}}{e^{\eta\Phi(\mathbf{s},G)} + e^{\eta\Phi(\mathbf{s},G+ij)}} \xi.$$

This equality is trivially satisfied. A similar argument holds for the removal of a link with a transition from $\boldsymbol{\omega} = (\mathbf{s}, G)$ to $\boldsymbol{\omega}' = (\mathbf{s}, G - ij)$ where the detailed balance condition reads

$$\frac{1}{\mathscr{Z}\eta} e^{\eta(\Phi(\mathbf{s},G) - m \ln(\frac{\xi}{\lambda}))} \frac{e^{\eta\Phi(\mathbf{s},G-ij)}}{e^{\eta\Phi(\mathbf{s},G-ij)} + e^{\eta\Phi(\mathbf{s},G)}} \xi = \frac{1}{\mathscr{Z}\eta} e^{\eta(\Phi(\mathbf{s},G-ij) - (m-1) \ln(\frac{\xi}{\lambda}))} \frac{e^{\eta\Phi(\mathbf{s},G)}}{e^{\eta\Phi(\mathbf{s},G)} + e^{\eta\Phi(\mathbf{s},G-ij)}} \lambda.$$

For a change in the agents' actions with a transition from $\boldsymbol{\omega} = (s_i, \mathbf{s}_{-i}, G)$ to $\boldsymbol{\omega}' = (s_i', \mathbf{s}_{-i}, G)$ we get the following detailed balance condition

$$\begin{aligned} &\frac{1}{\mathscr{Z}\eta} e^{\eta(\Phi(s_i,\mathbf{s}_{-i},G) - m \ln(\frac{\xi}{\lambda}))} \frac{e^{\eta\Phi(s_i',\mathbf{s}_{-i},G)}}{e^{\eta\Phi(s_i,\mathbf{s}_{-i},G)} + e^{\eta\Phi(s_i',\mathbf{s}_{-i},G)}} \chi \\ &\qquad = \frac{1}{\mathscr{Z}\eta} e^{\eta(\Phi(s_i',\mathbf{s}_{-i},G) - m \ln(\frac{\xi}{\lambda}))} \frac{e^{\eta\Phi(s_i,\mathbf{s}_{-i},G)}}{e^{\eta\Phi(s_i,\mathbf{s}_{-i},G)} + e^{\eta\Phi(s_i',\mathbf{s}_{-i},G)}} \chi. \end{aligned}$$

Hence, the probability measure $\mu^\eta(\boldsymbol{\omega})$ satisfies a detailed balance condition of Equation (26) and therefore is the stationary distribution of the Markov chain with transition rates $q(\boldsymbol{\omega}, \boldsymbol{\omega}')$. $\square$

Proof of Proposition 3. Assume w.l.o.g. that the agents are ordered such that $\gamma_1 = \ldots \gamma_{n_+} = +1$ and $\gamma_{n_+ + 1} = \ldots \gamma_n = -1$, with $0 \leq n_+ \leq n$. Let us consider all configurations $\mathbf{s} \in \{-1, +1\}^n$ for which there $k = 0, \ldots, n$ agents with $s_i = \gamma_i$. For a given k, there are $\binom{n_+}{j}$ ways to select j agents from n_+ choosing $s_i = \gamma_i = +1$, and there are $\binom{n-n_+}{k-j}$ ways to select $k - j$ agents from n_- choosing $s_i = \gamma_i = -1$, for each $j = 0, \ldots, \min\{k, n_+\}$. Hence, there are

$$\sum_{j=0}^{\min\{k,n_+\}} \binom{n_+}{j} \binom{n - n_+}{k - j}$$

ways to obtain alignments of $\boldsymbol{\gamma}$ and $\mathbf{s}$ such that $\sum_{i=1}^n s_i\gamma_i = k-(n-k) = 2k-n$.

Next, we consider the products $s_i s_j$. Since all the j agents in n_+ with $s_i=+1$ choose the same action $+1$, and all the $k-j$ agents in n_- with $s_i=-1$ choose the same action -1 we obtain

$$l(k,j) = \binom{j}{2} + \binom{k-j}{2} + \binom{n_+-j}{2} + \binom{n-n_+-(k-j)}{2} + (n_+-j)(k-j) + j(n-n_+-(k-j))$$

pairs whose product of actions gives $s_i s_j = +1$. The first term in the equation above counts all pairs of agents with action $+1$ in the first set (with all $\gamma_i=+1$), the second all pairs of agents with action -1 in the second set (with all $\gamma_i=-1$), the third term the pairs of agents with action -1 in the first set (with all $\gamma_i=+1$), the fourth term the pairs of agents with action $+1$ in the second set (with all $\gamma_i=-1$), the fifth term counts the pairs with agents in the first set who choose action -1 and the agents in the second set who chose action -1, while the last term counts the pairs with agents in the first set who choose action $+1$ and agents in the second set who also choose action $+1$.

We can further simplify $l(k,j)$ to

$$l(k,j) = \frac{n^2 + (2(2j-k)-1)n + 2(2j-k)^2 - 2(n+2(2j-k)-n_+)n_+}{2}.$$

Then we can write

$$\begin{aligned}
\mathscr{Z}^\eta &= \sum_{\mathbf{s}\in\{-1,+1\}^n} e^{\eta\mathscr{H}^\eta(\mathbf{s})} \\
&= \sum_{k=0}^{n} \sum_{j=0}^{\min\{k,n_+\}} \binom{n_+}{j}\binom{n-n_+}{k-j} \exp\Bigg\{ \eta(1-\theta)(2k-n) \\
&+ \frac{l(k,j)}{\eta}\ln\left(1+e^{\eta(\theta-\zeta)}\right) + \frac{\binom{n}{2}-l(k,j)}{\eta}\ln\left(1+e^{-\eta(\theta+\zeta)}\right)\Bigg\} \\
&= \sum_{k=0}^{n} \sum_{j=0}^{\min\{k,n_+\}} \binom{n_+}{j}\binom{n-n_+}{k-j} e^{\eta(1-\theta)(2k-n)} \left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}} \left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}},
\end{aligned}$$

where $n_+ = \#(\{\gamma_i = 1 : i = 1,\ldots,n\})$. □

Proof of Proposition 4. Knowing the partition function allows us to compute the expected number of links, m, as

$$\mathbb{E}^\eta(m) = \sum_{G\in\mathcal{G}^n}\sum_{\mathbf{s}\in\{-1,+1\}^n} m\mu^\eta(\mathbf{s},G) = \frac{1}{\mathscr{Z}^\eta}\sum_{G\in\mathcal{G}^n}\sum_{\mathbf{s}\in\{-1,+1\}^n} \underbrace{me^{\eta\Phi(\mathbf{s},G)}}_{-\frac{1}{\eta}\frac{\partial}{\partial\zeta}e^{\eta\Phi(\mathbf{s},G)}} = -\frac{1}{\eta}\frac{1}{\mathscr{Z}^\eta}\frac{\partial\mathscr{Z}^\eta}{\partial\zeta}. \quad (27)$$

With Equations (5) and (27) we then can compute the expected number of links as

$$\begin{aligned}
\mathbb{E}^\eta(m) = \frac{1}{\eta}\frac{1}{\mathscr{Z}^\eta}\sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}} \binom{n_+}{j}\binom{n-n_+}{k-j} e^{\eta(1-\theta)(2k-n)} \\
\times \left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}} \left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}} \left(\frac{l(k,j)}{1+e^{-\eta(\theta-\zeta)}} + \frac{\binom{n}{2}-l(k,j)}{1+e^{\eta(\theta+\zeta)}}\right).
\end{aligned}$$

For $\theta = 0$ this simplifies to

$$
\begin{aligned}
\mathbb{E}^\eta(m) &= \frac{1}{\eta}\frac{1}{\mathscr{Z}^\eta}\sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}}\binom{n_+}{j}\binom{n-n_+}{k-j}e^{\eta(2k-n)} \\
&\times \left(1+e^{-\eta\zeta}\right)^{\frac{l(k,j)}{\eta}}\left(1+e^{-\eta\zeta}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}}\left(\frac{l(k,j)}{1+e^{\eta\zeta}}+\frac{\binom{n}{2}-l(k,j)}{1+e^{\eta\zeta}}\right) \\
&= \frac{1}{\eta}\frac{1}{\mathscr{Z}^\eta}\sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}}\binom{n_+}{j}\binom{n-n_+}{k-j}\binom{n}{2}e^{\eta(2k-n)}\left(1+e^{-\eta\zeta}\right)^{\frac{\binom{n}{2}}{\eta}}\frac{1}{1+e^{\eta\zeta}} \\
&= \frac{1}{\eta}\frac{1}{\mathscr{Z}^\eta}\binom{n}{2}\left(1+e^{-\eta\zeta}\right)^{\frac{\binom{n}{2}}{\eta}}\frac{1}{1+e^{\eta\zeta}}\sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}}\binom{n_+}{j}\binom{n-n_+}{k-j}e^{\eta(2k-n)} \\
&= \frac{1}{\mathscr{Z}^\eta}\frac{e^{-\eta n}}{\eta\pi(1+e^{\eta\zeta})}\binom{n}{2}\left(1+e^{-\eta\zeta}\right)^{\frac{\binom{n}{2}}{\eta}} \\
&\times \left(\pi\left(1+e^{2\eta}\right)^n - e^{2(n+1)\eta}\sin(n\pi)\Gamma(n+1)\,{}_2F_1\left(1,1;n+2;-e^{2\eta}\right)\right),
\end{aligned}
$$

and one can show that for $\zeta > 0$ we have that $\lim_{\eta\to\infty}\mathbb{E}^\eta(m) = 0$. ◻

Proof of Proposition 15. For the average action level $\bar{s} = \frac{1}{n}\sum_{i=1}^n s_i = \frac{1}{n}\mathbf{u}^\top\mathbf{s}$ we have that

$$
\begin{aligned}
\mathbb{E}^\eta(\bar{s}) &= \sum_{\mathbf{s}\in\{-1,+1\}^n}\bar{s}\mu^\eta(\mathbf{s}) \\
&= \frac{1}{\mathscr{Z}^\eta}\sum_{\mathbf{s}\in\{-1,+1\}^n}\frac{1}{n}\mathbf{u}^\top\mathbf{s}e^{\eta\mathscr{H}^\eta(\mathbf{s})} \\
&= \frac{1}{\mathscr{Z}^\eta}\sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}}\binom{n_+}{j}\binom{n-n_+}{k-j}\frac{j+(n-n_+-(k-j))-(n_+-j+(k-j))}{n} \\
&\times e^{\eta(1-\theta)(2k-n)}\left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}}\left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}} \\
&= \frac{1}{\mathscr{Z}^\eta}\sum_{k=0}^{n}\sum_{j=0}^{\min\{k,n_+\}}\binom{n_+}{j}\binom{n-n_+}{k-j}\frac{n+4j-2(n_++k)}{n} \\
&\times e^{\eta(1-\theta)(2k-n)}\left(1+e^{\eta(\theta-\zeta)}\right)^{\frac{l(k,j)}{\eta}}\left(1+e^{-\eta(\theta+\zeta)}\right)^{\frac{\binom{n}{2}-l(k,j)}{\eta}}.
\end{aligned}
$$

◻

Proof of Proposition 6. Recall that the potential function in Proposition 1 is given by

$$
\Phi(\mathbf{s},G) = (1-\theta)\sum_{i=1}^n\gamma_i s_i + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(\theta s_i s_j - \zeta).
$$

Observe that the term $\sum_{i=1}^n\gamma_i s_i$ is maximized over $s_i\in\{-1,+1\}$ for $s_i = \text{sign}(\gamma_i)$ for $i = 1,\ldots,n$. The term $\sum_{i=1}^n\sum_{j=1}^n a_{ij}s_i s_j$ is maximized over $s_i, s_j\in\{-1,+1\}$ for $a_{ij} = 1$ iff $s_i = s_j$. Similarly, the term $\sum_{i=1}^n\sum_{j=1}^n a_{ij}(\theta s_i s_j - \zeta)$ is maximized over $s_i, s_j\in\{-1,+1\}$ for

$a_{ij} = 1$ iff $s_i = s_j$ and $\zeta < \theta$, while $a_{ij} = 1$ otherwise. The latter implies that the network must be either complete, empty, or composed of two disconnected cliques.

First, consider two cliques, K_{n_+} and K_{n-n_+} of sizes n_+ and $n - n_+$, respectively, where the agents in K_{n_+} choose $s_i = \gamma_i = +1$, and the agents in K_{n-n_+} choose $s_i = \gamma_i = -1$. The potential function is then given by

$$\Phi(\mathbf{s}, K_{n_+} \cup K_{n-n_+}) = (1-\theta)n + \frac{1}{2}(n(n-1) - 2n_+(n-n_+))(\theta - \zeta).$$

Next, consider the complete graph K_n in which all agents choose $s_i = +1$. Then

$$\Phi(\mathbf{s}, K_n) = (1-\theta)(2n_+ - n) + \frac{n(n-1)}{2}(\theta - \zeta).$$

Similarly, in the complete graph K_n in which all agents choose $s_i = -1$, the potential is given by

$$\Phi(\mathbf{s}, K_n) = (1-\theta)(n - 2n_+) + \frac{n(n-1)}{2}(\theta - \zeta).$$

The potential in the complete graph K_n is then larger than in the union of cliques, $K_{n_+} \cup K_{n-n_+}$, if

$$(1-\theta)n - n_+(n-n_+)(\theta-\zeta) < (1-\theta)\max\{2n_+ - n, n - 2n_+\}.$$

Solving for θ yields

$$\theta > \frac{n + n_+(n-n_+)\zeta - \max\{2n_+ - n, n - 2n_+\}}{n(n_+ + 1) - n_+^2 - \max\{2n_+ - n, n - 2n_+\}}. \tag{28}$$

Finally, note that the potential in any union of cliques $K_{n_1} \cup K_{n_2}$ with $n_1 + n_2 = n$ is smaller than the potential in the union of cliques $K_{n_+} \cup K_{n-n_+}$ or the complete graph K_n. To see this consider disconnecting a node j from the clique K_{n_+} and connecting it to all nodes in the clique K_{n-n_+}, while choosing the action $s_j = -1$ with $\gamma_j = +1$. This is illustrated in Figure 2. W.l.o.g. assume that $n_+ < \frac{n}{2}$. The resulting change in the potential is given by

$$\Phi(\mathbf{s}', K_{n_+-1} \cup K_{n-n_++1}) - \Phi(\mathbf{s}, K_{n_+} \cup K_{n-n_+}) = ((n-n_+) - (n_+ - 1))(\theta - \zeta) - 2(1-\theta).$$

The second term in the above equation comes from the loss of agent j choosing an action $s_j = -1$ that is not aligned with her preference $\gamma_j = +1$. The first term comes from the gain of having more pairs of connected agents with identical actions. Now, when repeating this procedure and attaching another node to K_{n-n_+}, then the loss term will remain the same while the gain term will increase because there are more agents with the same action in the larger clique to connect to. Hence, when all nodes have been reattached, in this way, we obtain a complete graph K_n with a potential that is larger than the potential obtained for the network $K_{n_+-1} \cup K_{n-n_++1}$. This shows that the potential must be largest either in the network composed of two cliques, $K_{n_+-1} \cup K_{n-n_++1}$, in which all agents choose actions according to their idiosyncratic preferences, $\gamma_i = s_i$, or in the complete graph K_n where all agents choose the same action.

□

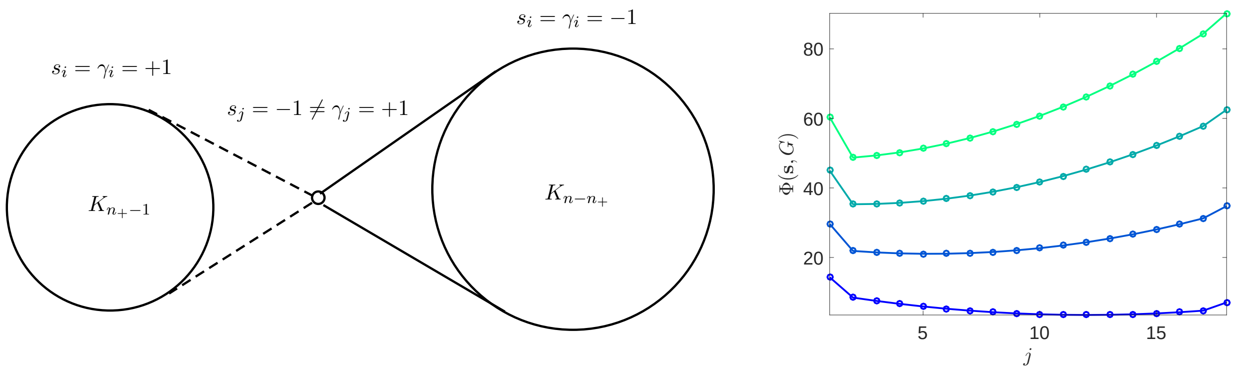

Figure 2: (Left panel) Illustration of two cliques, K_{n+} and K_{n-n_+} and the relocation of one node j from K_{n+} to K_{n-n_+}. (Right panel) The resulting potential for relocating node j from the clique K_{n_+} to the clique K_{n-n_+} for $\theta \in \{0.050.0750.10.125\}$, $n_+ = 17$, $n = 50$ and $\zeta = 0.1$. For small values of θ the union of cliques $K_{n_+} \cup K_{n-n_+}$ ($j = 0$) has the highest potential, while for increasing values of θ the potential is highest for the complete graph K_n ($j = n_+ = 17$). We also see that the potential in a union of cliques $K_{n_+-k} \cup K_{n-n_++k}$ for $k = 1, \ldots, n_+ - 1$ is always smaller than the potential in the complete graph K_n or in the union of cliques $K_{n_+} \cup K_{n-n_+}$.